Mondriaan partitioning for faster parallel integer factorisation

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Outline

1. Attacking cryptosystems:
   - integer factorisation attack on RSA
   - sparse binary matrix
   - block Lanczos algorithm
2. Mondriaan partitioning
   - sparse matrix–vector multiplication
   - matrix partitioning (joint with Brendan Vastenhouw)
   - vector partitioning (joint with Wouter Meesen)
3. Experimental results
4. Another application: PageRank
   - Ranking web pages (joint with Tristan van Leeuwen, Ümit Çatalyürek)
5. Conclusions and future work

Cracking RSA

- RSA cryptosystem is based on difficulty of integer factorisation.
- Aim: given large n, find primes p, q such that $pq = n$.
- Recent record: May 9, 2005. RSA-200 with 200 decimal digits by Bahr, Böhm, Franke, Kleinjung.
- 55 CPU years of sieving (on 2.2 GHz Opterons) gives many pairs (a, b) with $a \equiv b \pmod{n}$. Each a and b is composed of small primes.
- Example for $n = 33$:
  $a_1 = 2^2 \cdot 7$, $b_1 = -1 \cdot 5$
  $a_2 = 7^3$, $b_2 = -1 \cdot 2^2 \cdot 5$.
- Note: $a_1 \cdot a_2 = (2 \cdot 7^2)^2$ and $b_1 \cdot b_2 = (-1 \cdot 2 \cdot 5)^2$.

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Solving sparse linear systems in GF(2)

- In general, desired subset S of pairs (a_j, b_j) such that $\prod_{j\in S} a_j$ and $\prod_{j\in S} b_j$ are both square.
- Translate into linear algebra. Bitmatrix A:
  a_{ij} = exponent of prime p_i in a_j (mod 2), where p_i is the ith prime, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$, etc. and $p_0 = -1$.
- A is sparse, since not all primes are represented in an a_j.
- $A\mathbf{x}$ is linear combination of columns in A.
  Solving $A\mathbf{x} = 0$ in GF(2) gives S = $\{j : x_j = 1\}$.

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Example: matrix A

a	25	32	1	28	40	35	2560	128	125	343
$p = 2$	0	5	0	2	3	0	9	7	0	0
$p = 5$	2	0	0	0	1	1	1	0	3	0
$p = 7$	0	0	0	1	0	1	0	0	0	3

Take the entries modulo 2:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example: matrix C

- Also generate B. Solve $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$ together. Let $C\mathbf{x} = 0$ be the larger simultaneous system:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

- For RSA-200, solution took 3 months on a cluster of 80 processors. Sparse matrix C has 64 million rows and columns and 11×10^9 nonzeros.

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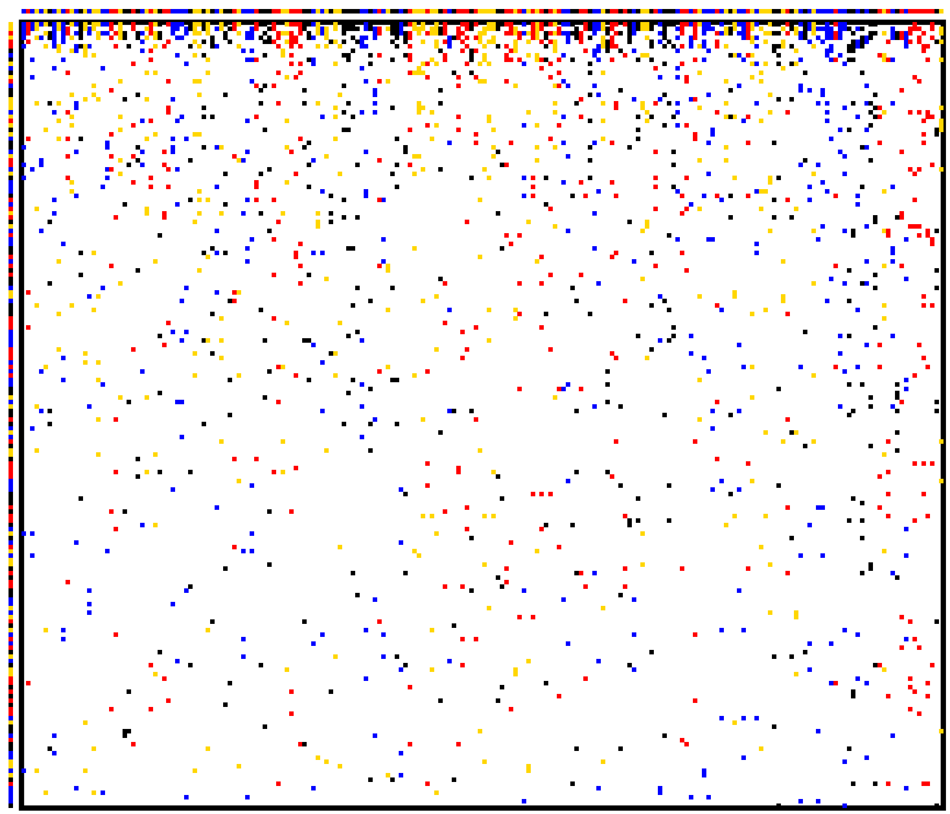
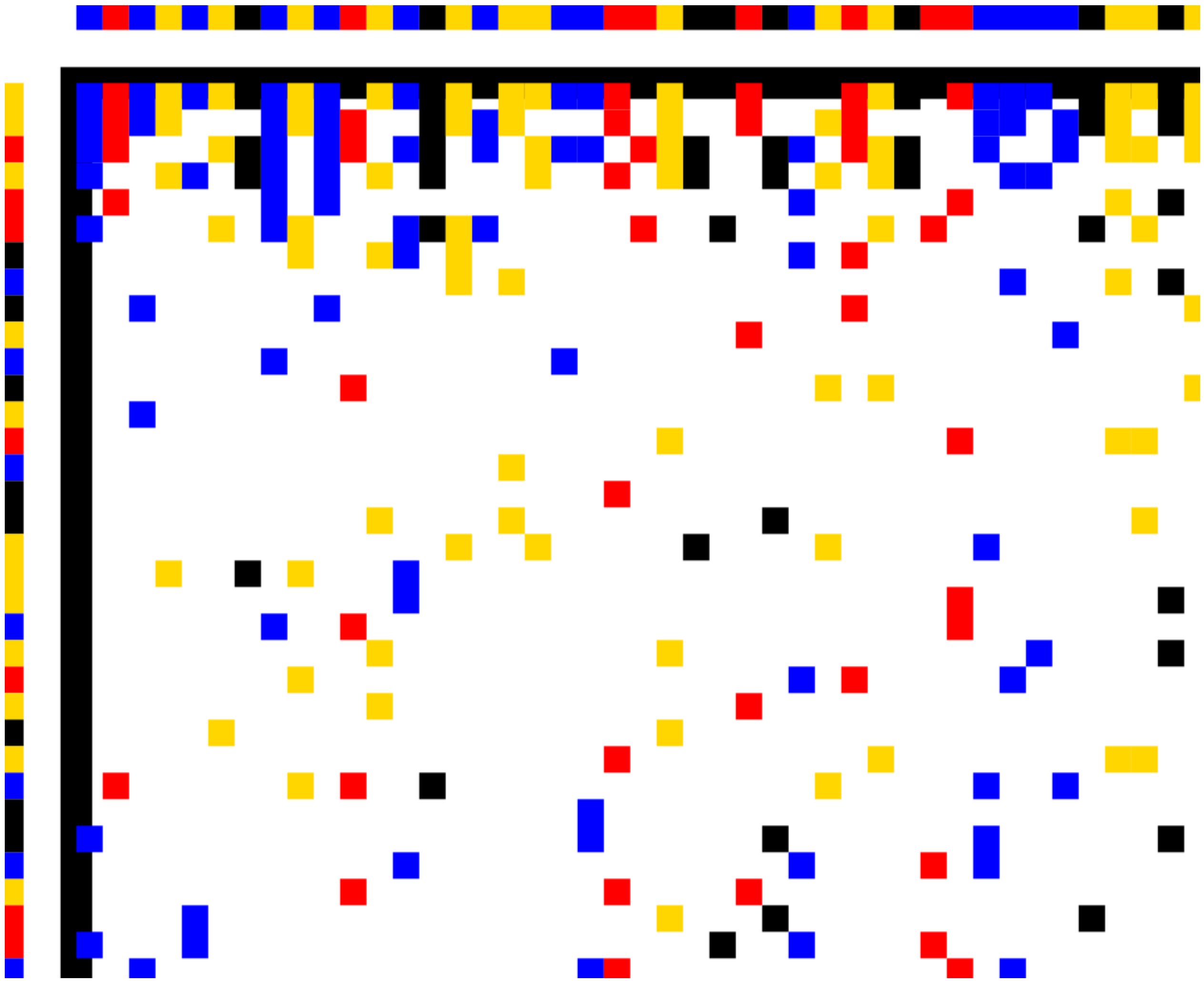
Quadratic sieving matrix `MPQS30`

Size 210×179, 1916 nonzeros, 30 decimal digits.
Partitioned for 4 processors (red, black, blue, orange) by the Mondriaan package



Matrix: courtesy of Richard Brent, 2001

Left upper corner of *MPQS30*



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Block Lanczos algorithm by Montgomery (1995)

- Use Lanczos for symmetric systems: solve $C^T C\mathbf{x} = 0$.
- Find 32 different solutions by the block Lanczos algorithm, solving a system $C^T C X = 0$.
  X has 32 columns (word size of the computer) and can be viewed as an integer vector.
- C and C^T are not explicitly multiplied.
  Only C is stored: rectangular sparse bitmatrix.

Main loop of block Lanczos algorithm

input: C = sparse $n_1 \times n_2$ bitmatrix,
Y = dense random $n_2 \times 32$ bitmatrix.
output: X = dense $n_2 \times 32$ bitmatrix such that $C^T C X = C^T C Y$.
while $Cond_i \neq 0$ **do**

$[W_i^{\text{inv}}, SS_i^T] = \ldots;$ $\{32 \times 32\}$
$X = X + V_i * (W_i^{\text{inv}} * (V_i^T * V_0));$
$C^T CV_i = C^T \circledast CV_i;$ {matvec}
$K_i = (V^T C_i^T * (C \circledast (C^T CV_i))) * SS_i^T + Cond_i;$

$D_{i+1}, E_{i+1}, F_{i+1} = \ldots;$ $\{32 \times 32\}$
$V_{i+1} = C^T CV_i * SS_i^T + V_i * D_{i+1} + V_{i-1} * E_{i+1} + V_{i-2} * F_{i+1}$
$V^T C_{i+1}^T = V_{i+1}^T \circledast C^T;$
$CV_{i+1} = C \circledast V_{i+1};$
$Cond_{i+1} = V^T C_{i+1}^T * CV_{i+1};$
$i = i + 1$

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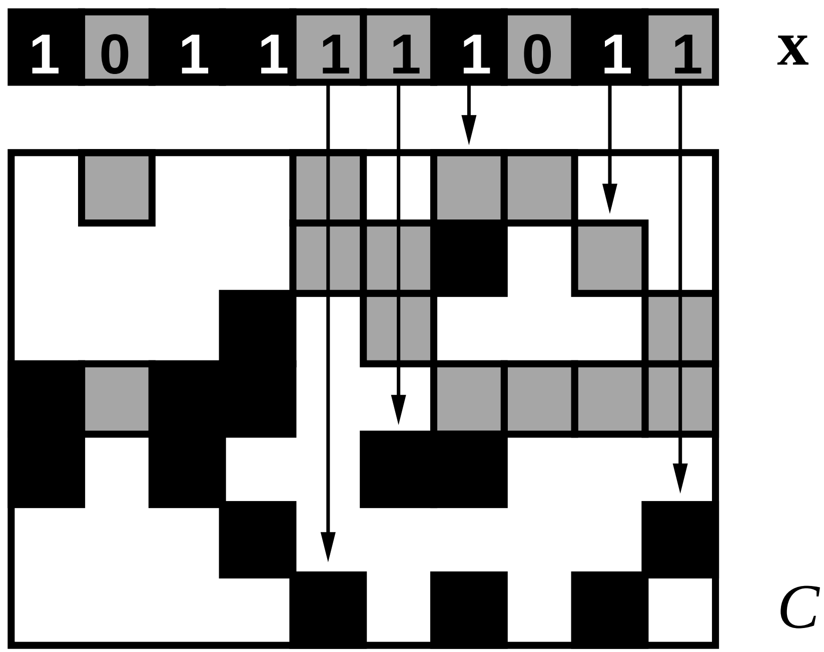
Parallel sparse matrix–vector multiplication $\mathbf{y} := C\mathbf{x}$

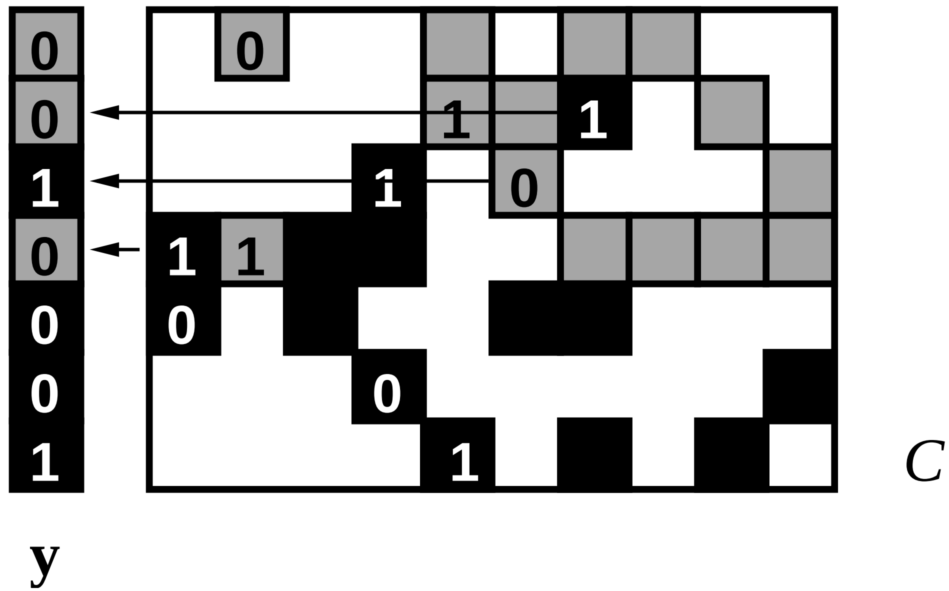
C sparse $n_1 \times n_2$ matrix, $\mathbf{y}$ dense n_1-vector, $\mathbf{x}$ dense n_2-vector

$$y_i := \sum_{j=0}^{n_2-1} a_{ij} x_j$$



Vertical communication. $p = 2$

Parallel sparse matrix–vector multiplication (cont'd)



Horizontal communication. $p = 2$

- Algorithm has 4 supersteps: communicate, compute, communicate, compute

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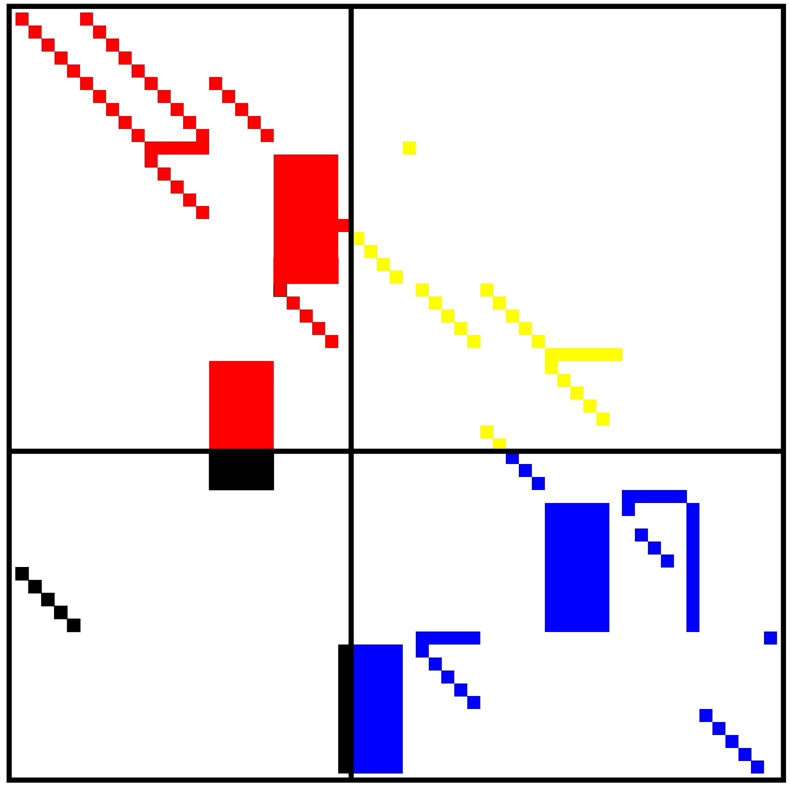
Cartesian matrix partitioning



- Block distribution of 59×59 matrix `impcol_b` with 312 nonzeros, for $p = 4$
- #nonzeros per processor: 126, 28, 128, 30

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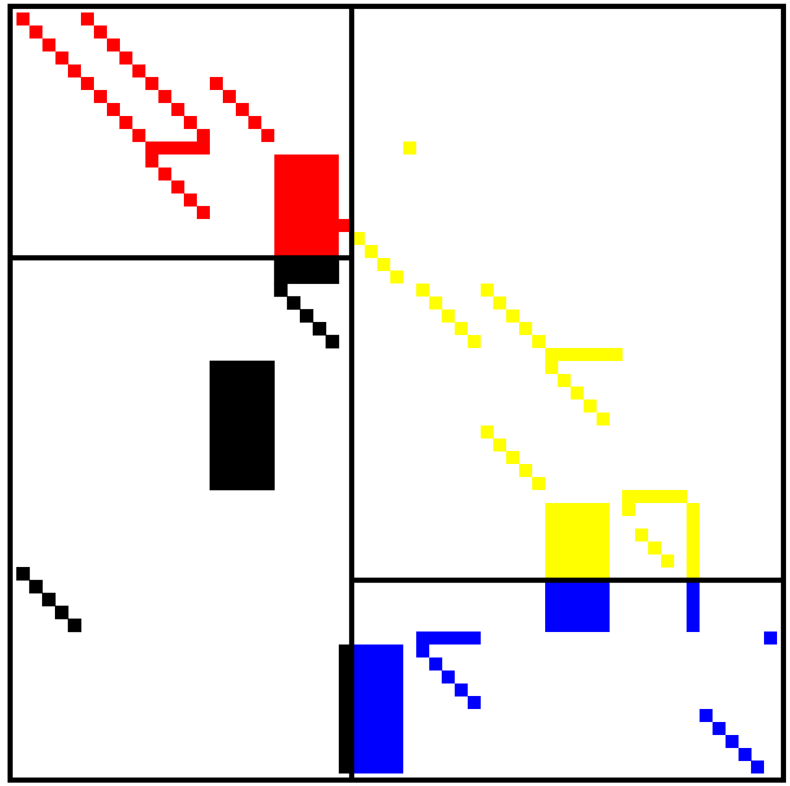
Non-Cartesian matrix partitioning



- Block distribution of 59×59 matrix `impcol_b` with 312 nonzeros, for $p = 4$
- #nonzeros per processor: 76, 76, 80, 80

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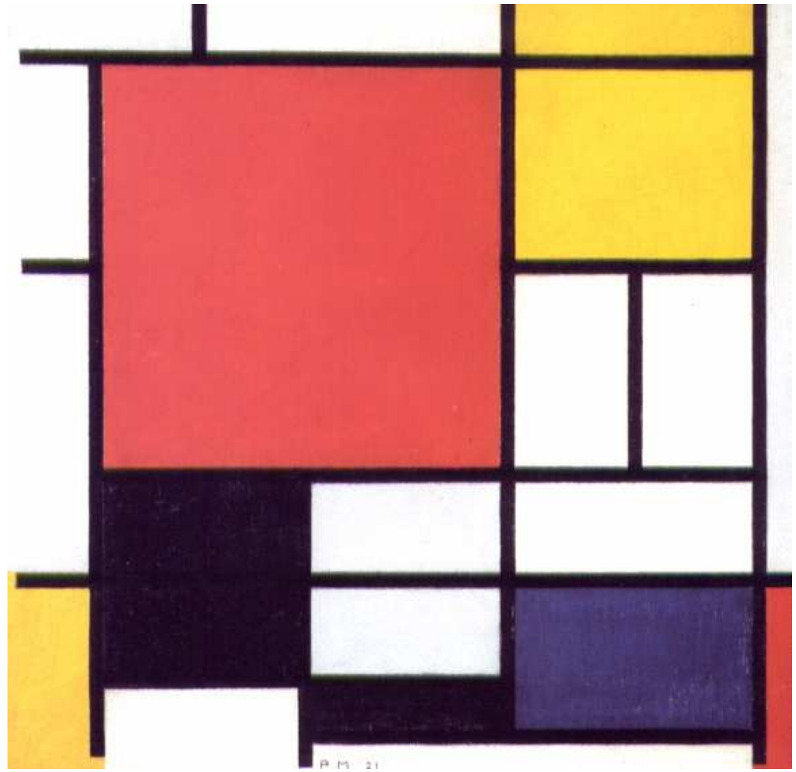
Composition with Red, Yellow, Blue and Black



Piet Mondriaan 1921

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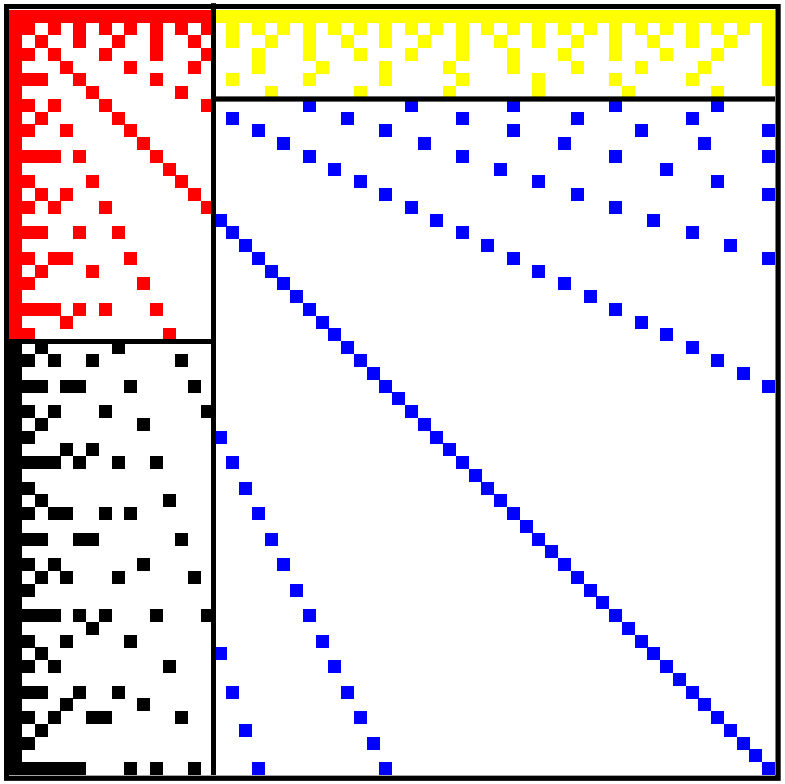
Mondriaan painted here

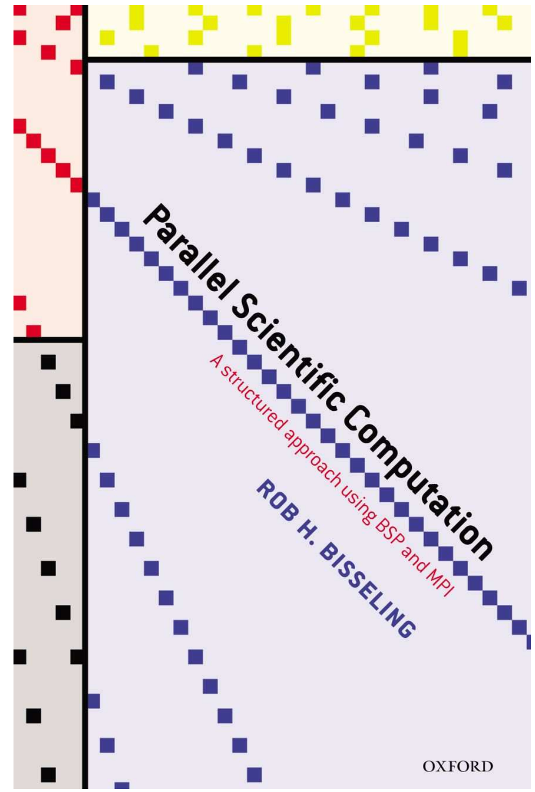
Richard, Erin, Rona, Sarai (Abcoude, NL, 2001)

Mill in Sunlight

Piet Mondriaan 1908

Matrix `prime60`





- Block distribution of 60×60 matrix `prime60` with 462 nonzeros, for $p = 4$
- $a_{ij} \neq 0 \Longleftrightarrow i|j \text{ or } j|i \qquad (1 \leq i, j \leq 60)$

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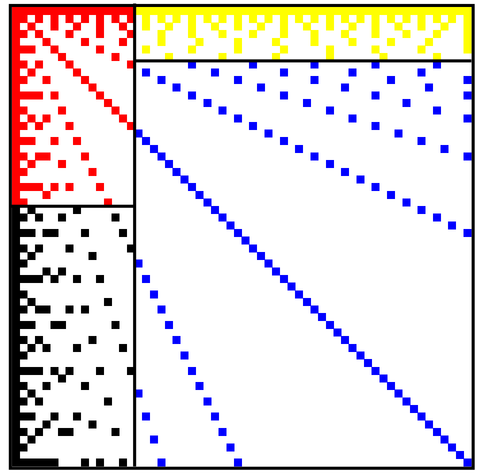
Communication volume for partitioned matrix



$$V(A_0, A_1, A_2, A_3) = V(A_0, A_1, A_2 \cup A_3) + V(A_2, A_3)$$

Here, $V(A_0, A_1, A_2, A_3)$ is the global matrix–vector communication volume corresponding to the partitioning A_0, A_1, A_2, A_3

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Recursive, adaptive bipartitioning algorithm

MatrixPartition(A, p, ϵ)
input: ϵ = allowed load imbalance, $\epsilon > 0$.
output: p-way partitioning of A with imbalance $\leq \epsilon$.

if $p > 1$ **then**

$q := \log_2 p$;
$(A_0^{\mathrm{r}}, A_1^{\mathrm{r}}) := h(A, \mathrm{row}, \epsilon/q)$; hypergraph splitting
$(A_0^{\mathrm{c}}, A_1^{\mathrm{c}}) := h(A, \mathrm{col}, \epsilon/q)$;
if $V(A_0^{\mathrm{r}}, A_1^{\mathrm{r}}) \leq V(A_0^{\mathrm{c}}, A_1^{\mathrm{c}})$ **then**
$(A_0, A_1) := (A_0^{\mathrm{r}}, A_1^{\mathrm{r}})$
else $(A_0, A_1) := (A_0^{\mathrm{c}}, A_1^{\mathrm{c}})$

$maxnz := \frac{nz(A)}{p}(1+\epsilon)$;
$\epsilon_0 := \frac{maxnz}{nz(A_0)} \cdot \frac{p}{2} - 1$; MatrixPartition($A_0, p/2, \epsilon_0$);
$\epsilon_1 := \frac{maxnz}{nz(A_1)} \cdot \frac{p}{2} - 1$; MatrixPartition($A_1, p/2, \epsilon_1$);

else output A;

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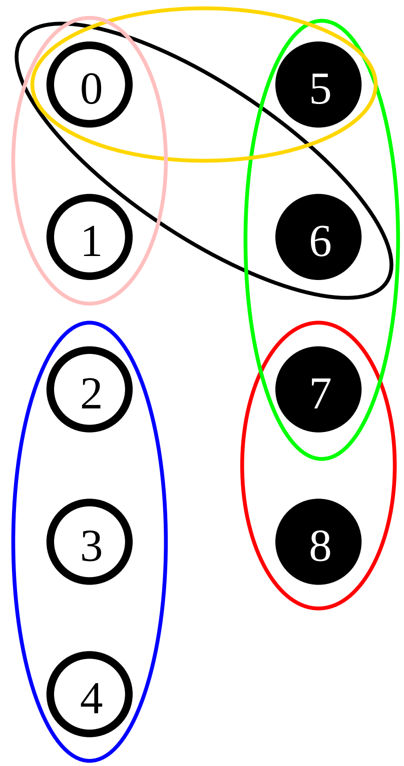
Hypergraph



Hypergraph with 9 vertices and 6 hyperedges (nets), partitioned over 2 processors

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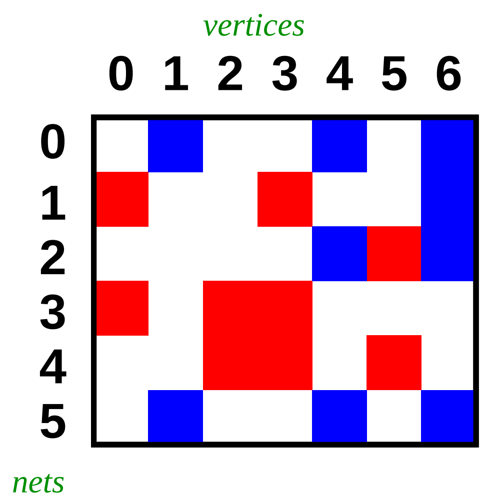
The h-function

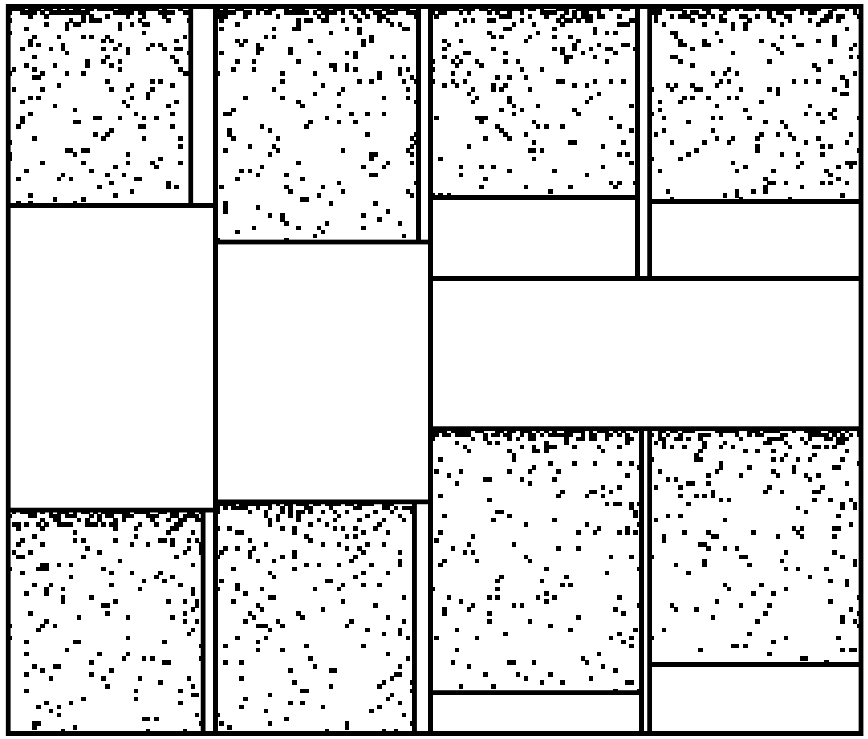


Column bipartitioning of $m \times n$ matrix

- Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{N}) \Rightarrow$ exact communication volume.
- Columns $\equiv$ Vertices: $0, 1, 2, 3, 4, 5, 6$.
  Rows $\equiv$ Hyperedges (nets, subsets of $\mathcal{V}$):

$$n_0 = \{1, 4, 6\}, \quad n_1 = \{0, 3, 6\}, \quad n_2 = \{4, 5, 6\},$$
$$n_3 = \{0, 2, 3\}, \quad n_4 = \{2, 3, 5\}, \quad n_5 = \{1, 4, 6\}.$$

Local view of Mondriaan distribution for 8 processors



- Imbalance $\epsilon = 3\%$
- First split is vertical
- Empty blocks collect empty row/column parts

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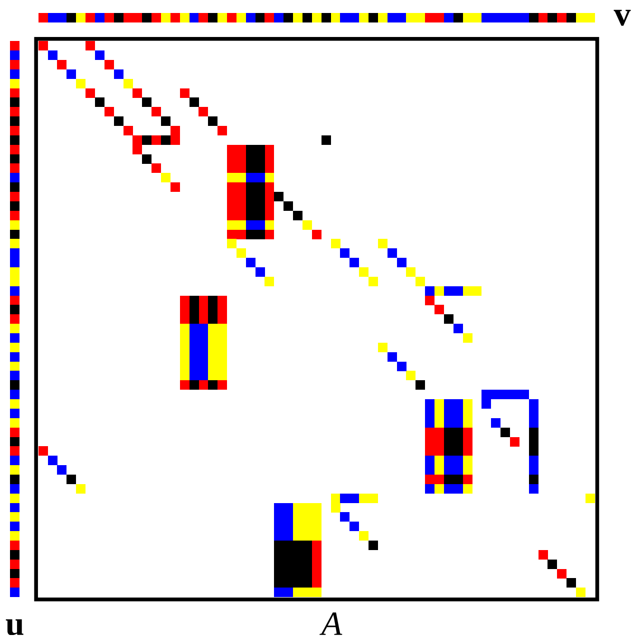
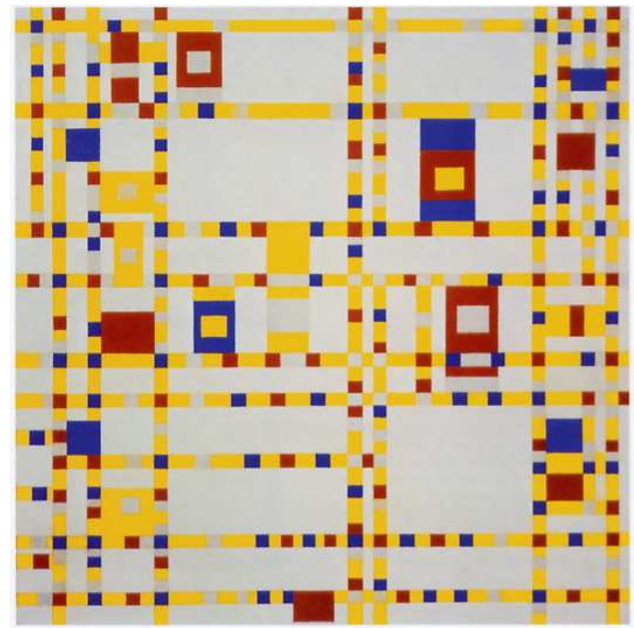
Vector partitioning





Broadway Boogie Woogie, Piet Mondriaan 1943

- No extra communication if:
  $v_j \mapsto$ one of the owners of a nonzero in matrix column j
  $u_i \mapsto$ owner in matrix row i
- This creates a separate vector partitioning problem.

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Balance the communication!

Reduce the cost by the bulk synchronous parallel (BSP) model

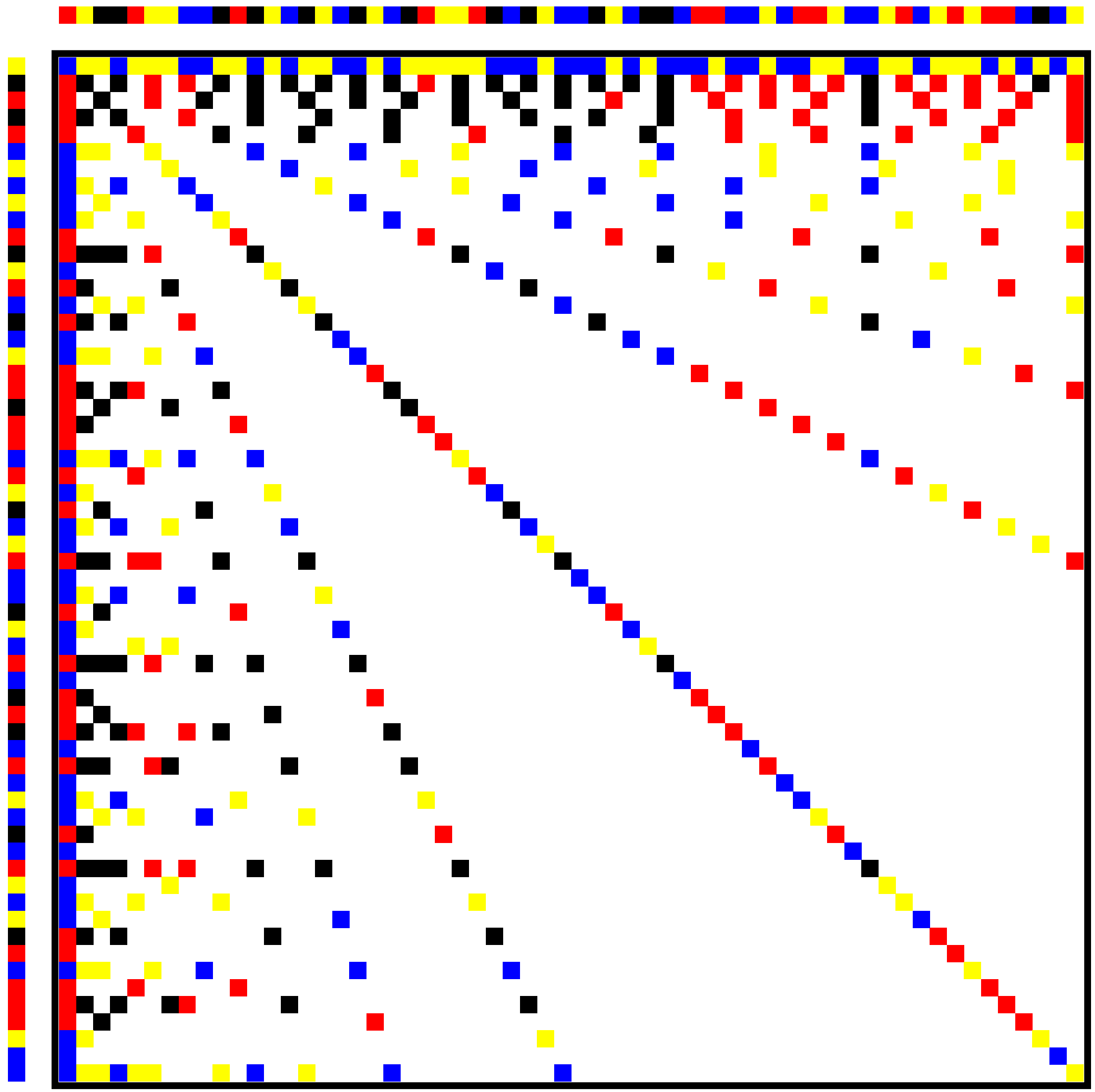
$$\max_{0 \le s < p} h(s),$$

where

$$h(s) = \max(h_{\mathrm{send}}(s), h_{\mathrm{recv}}(s))$$

for processor s

Vector partitioning for `prime60`



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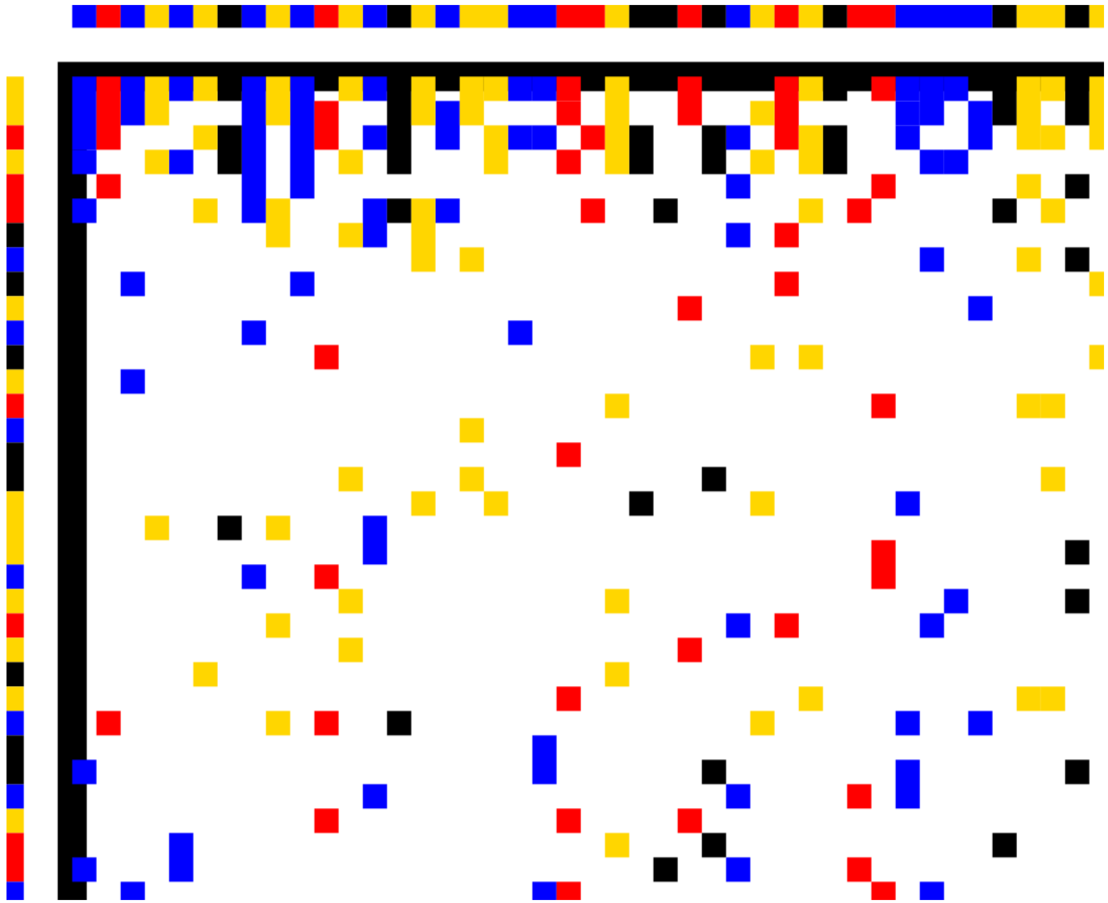
Vector partitioning for `MPQS30`



One-dimensional column partitioning of matrix fixes input vector partitioning. Much freedom for output vector.

Vector partitioning for parallel block Lanczos

- Matrix C, vector X, and vector $Y = CX$ are distributed by Mondriaan.
- C^T multiplication is reverse of C: swap input/output vectors, sends/receives.
- $C^T * C * X$: Output of C = Input of C^T. Hence: independent vector distributions, full freedom for communication balancing.

Vector inner products

- $V^T * V$ with V an $n_2 \times 32$ bitmatrix, i.e., an integer vector of length n_2.
- Easy if all vectors of the same length are partitioned in the same way.

Global-local indexing mechanism

- Processor owning matrix nonzero a_{ij} knows that it needs vector component x_j, but does not know where it is.
- Processor owning vector component x_j does not know where to send it.
- Solution: use a notice board (or data directory).
- x_j has global index j. Its address (its owner and local index) is first stored at a place that everyone can inspect, in processor $j \bmod p$ at location $j \operatorname{\mathbf{div}} p$.
- Before getting x_j, the owner of a_{ij} obtains its address in a preprocessing step.

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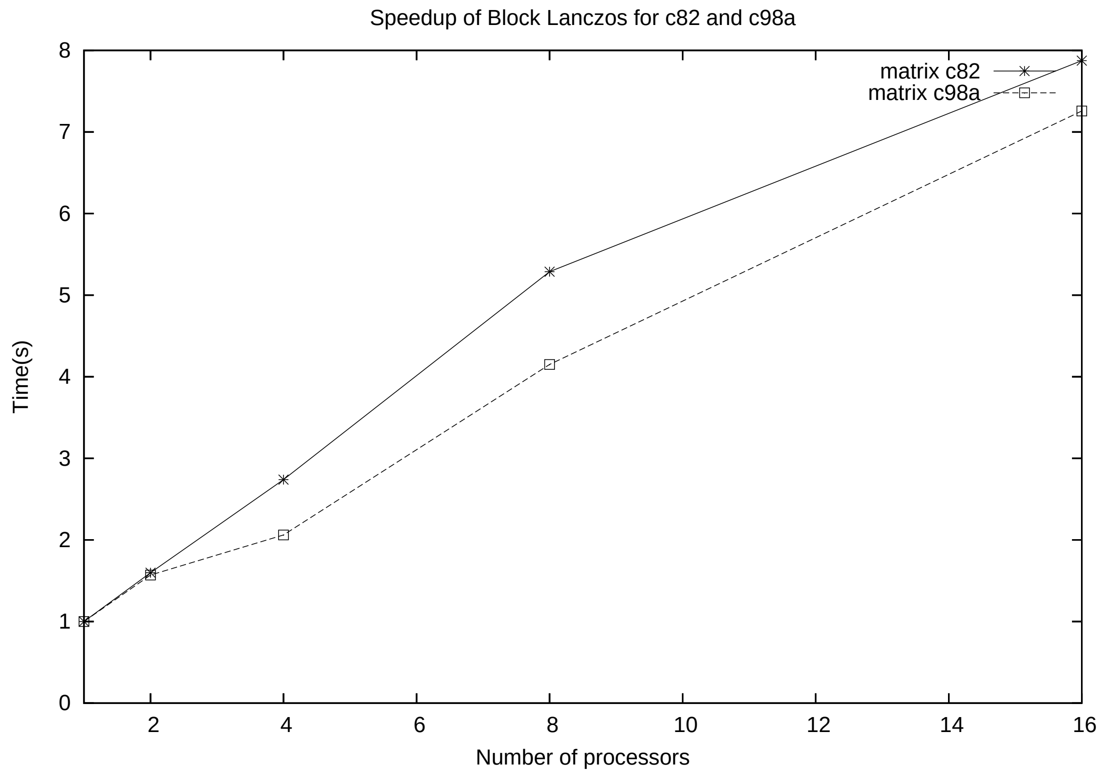
Experimental results on SGI Origin 3800



Name	n_1	n_2	$nz(C)$
c82	16307	16338	507716
c98a	56243	56274	2075889

Source: Richard Brent

Timings of main algorithm parts for matrix c82

p	Input	Lanczos	PP	Total
1	1.15	78.27	0.47	79.90
2	1.12	48.98	0.25	50.36
4	1.13	28.57	0.15	29.85
8	1.15	14.80	0.08	16.02
16	1.30	9.94	0.07	11.31

- Time (in s) of input phase, block Lanczos algorithm, postprocessing (PP), and total run time.
- Average over three runs.

Timings of main algorithm parts for matrix c98a

p	Input	Lanczos	PP	Total
1	4.1	1186.4	4.0	1194.5
2	4.0	755.8	1.9	761.7
4	3.9	575.5	0.6	580.0
8	4.0	285.8	0.5	290.3
16	4.1	163.5	0.2	167.8

BSP cost for Mondriaan partitioning of c82

p	Comp	Comm	Sync	V/p
1	1015432			
2	522926	$6277g$	l	6277
4	261462	$8078g$	$2l$	7154
8	130730	$7911g$	$2l$	5778
16	65366	$8298g$	$2l$	4296

Compare with cost for 2D square partitioning

$$
\begin{aligned}
T_{\mathrm{Matvec}} &\approx \frac{2nz(C)}{p} + \frac{n_1 + n_2}{\sqrt{p}} g + 2l \\
&= 63464 + 8161g + 2l \quad \text{for } \texttt{c82},\ p = 16.
\end{aligned}
$$

Here, g = time for communicating one data word;
l = global synchronisation time

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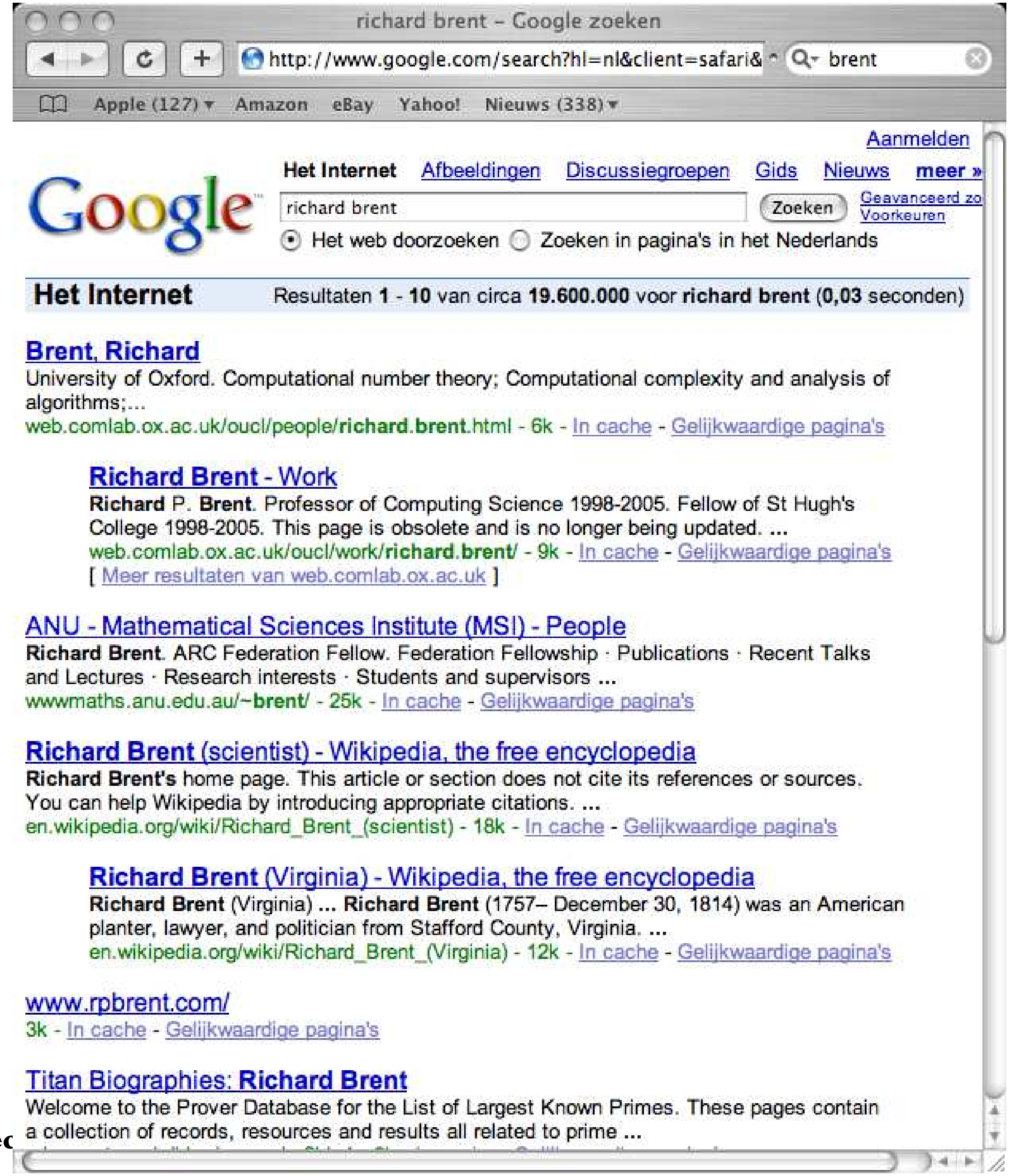
Web searching: which page ranks first?



The link matrix A

- Given n web pages with links between them.
  We can define the sparse $n \times n$ link matrix A by

$$a_{ij} = \begin{cases} 1 & \text{if there is a link from page } j \text{ to page } i \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\mathbf{e} = (1, 1, \ldots, 1)^T$, representing an initial uniform importance (rank) of all web pages. Then

$$(A\mathbf{e})_i = \sum_j a_{ij} e_j = \sum_j a_{ij}$$

  is the total number of links pointing to page i.
- The vector $A\mathbf{e}$ represents the importance of the pages; $A^2\mathbf{e}$ takes the importance of the pointing pages into account as well; and so on.

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The Google matrix

- A web surfer chooses each of the outgoing N_j links from page j with equal probability. Define the $n \times n$ diagonal matrix D with $d_{jj} = 1/N_j$.
- Let α be the probability that a surfer follows an outlink of the current page. Typically $\alpha = 0.85$. The surfer jumps to a random page with probability $1 - \alpha$.
- The Google matrix is defined by (Brin and Page 1998)

$$G = \alpha A D + (1 - \alpha)\mathbf{e}\mathbf{e}^T/n.$$

- The PageRank of a set of web pages is obtained by repeated multiplication by G, involving sparse matrix–vector multiplication by A, and some vector operations.

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Comparing 1D, 2D fine-grain, and 2D Mondriaan

- The following 1D and 2D fine-grain communication volumes for PageRank matrices are published results from the parallel program Par$k$way v2.1 (Bradley, de Jager, Knottenbelt, Trifunović 2005).
- The fine-grain method has been proposed by Çatalyürek and Aykanat in 2001.
- The 2D Mondriaan volumes are results with our recent improvements (to be incorporated in version 2.0), using only row/column partitioning, not the fine-grain option.

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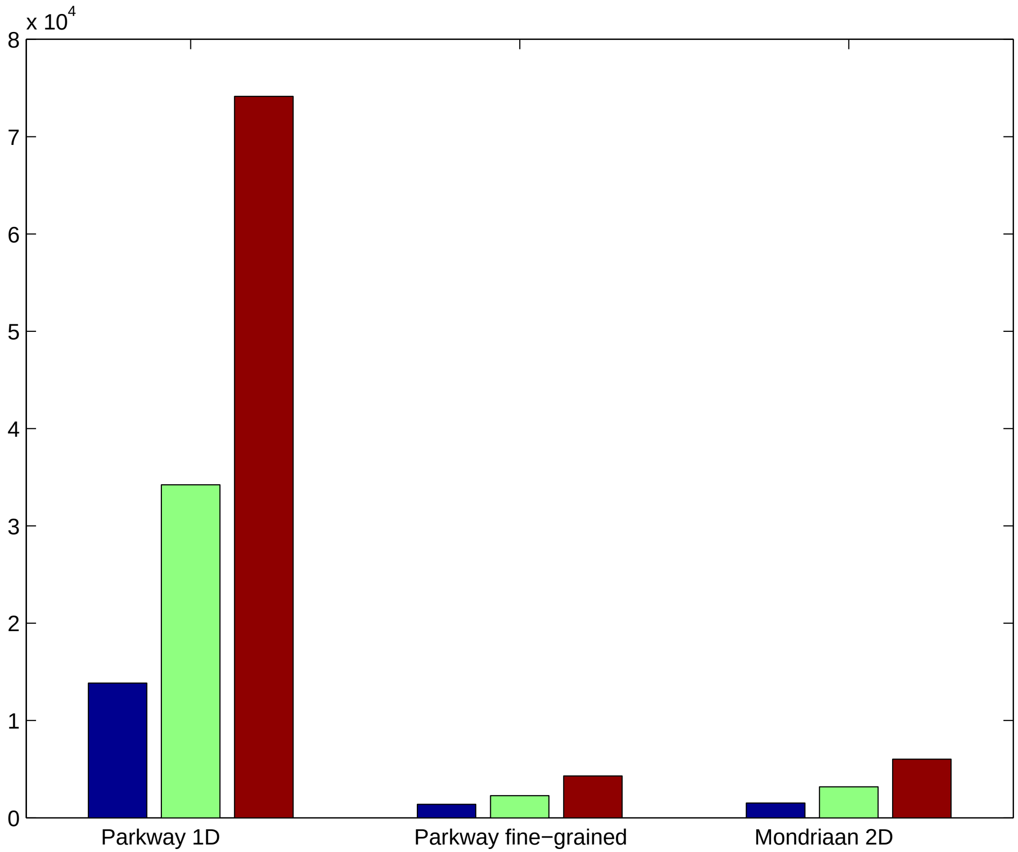
Communication volume: PageRank matrix Stanford



$p = 4, 8, 16$

- $n = 281,903$ (pages), $nz(A) = 2,594,228$ nonzeros (links).
- Represents the Stanford WWW subdomain, obtained by a web crawl in September 2002 by Sep Kamvar.

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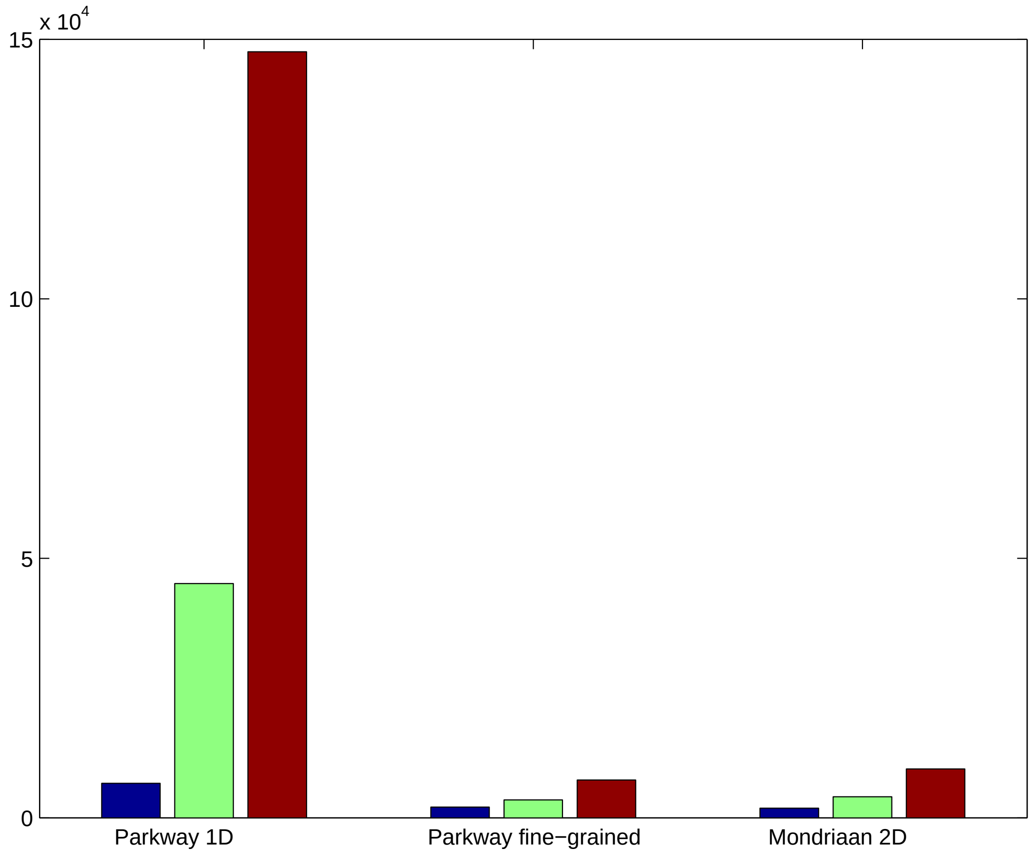
Communication volume: Stanford_Berkeley



$p = 4, 8, 16$

- $n = 683,446$, $nz(A) = 8,262,087$ nonzeros.
- Represents the Stanford and Berkeley subdomains, obtained by a web crawl in Dec. 2002 by Sep Kamvar.

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Meaning of PageRank results

- Both 2D methods save an order of magnitude in communication volume compared to 1D.
- Parkway fine-grain is slightly better than Mondriaan, in terms of partitioning quality. This may be due to a better implementation, or due to the fine-grain method itself. Further investigation is needed.
- 2D Mondriaan is much faster than fine-grain, since the hypergraphs involved are much smaller: 7×10^5 vs. 8×10^6 vertices for `Stanford_Berkeley`.

Conclusion

- We have identified 3 main building blocks for parallel integer factorisation:
  - sparse matrix–vector multiplication: most intensive computation
  - sparse matrix partitioning: reduces communication volume
  - vector partitioning: balances communication load
- Integer factorisation matrices remain a challenge for partitioners.
- Partitioning must be two-dimensional, both for integer factorisation and PageRank matrices.

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